ارتقاء مبهم‌سازی از $\mathbf{NC^1}$ به $\mathbf{P/poly}$ و رمزگذاری تابعی

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جنبه‌های ریاضی علوم کامپیوتری – مبانی رمز

Bootstrapping iO from $\mathbf{NC}^1$ to $\mathbf{P/poly}$ and Functional Encryption

Pooya Farshim

Queen's University Belfast/École Normale Supérieure

Mathematical Aspects of Computer Science – Foundations of Cryptography

Breakthrough Result of 2013

Candidate Indistinguishability Obfuscation and Functional Encryption for all circuits

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July 21, 2013

Abstract

In this work, we study *indistinguishability obfuscation* and *functional encryption* for general circuits:

Indistinguishability obfuscation requires that given any two equivalent circuits C_0 and C_1 of similar size, the obfuscations of C_0 and C_1 should be computationally indistinguishable.

In functional encryption, ciphertexts encrypt inputs x and keys are issued for circuits C. Using the key SK_C to decrypt a ciphertext $\mathrm{CT}_x = \mathsf{Enc}(x)$, yields the value $C(x)$ but does not reveal anything else about x. Furthermore, no collusion of secret key holders should be able to learn anything more than the union of what they can each learn individually.

We give constructions for indistinguishability obfuscation and functional encryption that supports all polynomial-size circuits. We accomplish this goal in three steps:

- We describe a candidate construction for indistinguishability obfuscation for $\mathbf{NC}^1$ circuits. The security of this construction is based on a new algebraic hardness assumption. The candidate and assumption use a simplified variant of multilinear maps, which we call *Multilinear Jigsaw Puzzles*.
- We show how to use indistinguishability obfuscation for $\mathbf{NC}^1$ together with Fully Homomorphic Encryption (with decryption in $\mathbf{NC}^1$) to achieve indistinguishability obfuscation for all circuits.
- Finally, we show how to use indistinguishability obfuscation for circuits, public-key encryption, and non-interactive zero knowledge to achieve functional encryption for all circuits. The functional encryption scheme we construct also enjoys succinct ciphertexts, which enables several other applications.

Indistinguishability Obfuscation (iO)

Security: Recall iO security:

$$\mathsf{iO}(C_0) \overset{c}{\approx} \mathsf{iO}(C_1) \ .$$

for **functionally equivalent** C_0 and C_1.

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$\underline{\mathrm{IND}^{\mathcal{S},\mathcal{A}}_{\mathsf{Obf}}(1^\lambda)}$:	$\underline{\mathrm{Eq}^{\mathcal{D}}_{\mathcal{S}}(1^\lambda)}$:
$b \leftarrow\$ \{0,1\}$	$(C_0, C_1, st) \leftarrow\$ \mathcal{S}(1^\lambda)$
$(C_0, C_1, st) \leftarrow\$ \mathcal{S}(1^\lambda)$	$x \leftarrow\$ \mathcal{D}(C_0, C_1, st)$
$C \leftarrow\$ \mathsf{Obf}(1^\lambda, C_b)$	return $(C_0(x) \neq C_1(x))$
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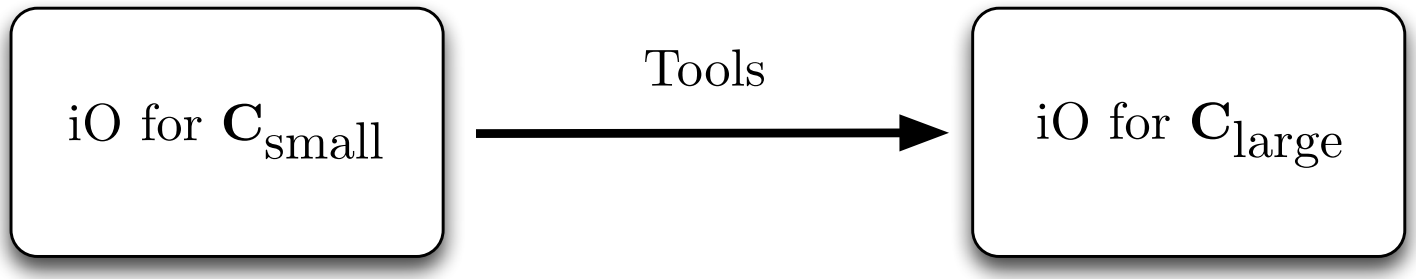
Sampler $\mathcal{S}$ is Eq-secure if for all (even unbounded) $\mathcal{D}$

$$\Pr\left[\mathrm{Eq}^{\mathcal{D}}_{\mathcal{S}}(1^\lambda)\right] \in \mathrm{Negl} \ .$$

Security for all Eq-secure ppt $\mathcal{S}$ and all ppt $\mathcal{A}$

$$\mathbf{Adv}^{\mathrm{ind}}_{\mathsf{Obf},\mathcal{S},\mathcal{A}}(\lambda) := 2 \cdot \Pr\left[\mathrm{IND}^{\mathcal{S},\mathcal{A}}_{\mathsf{Obf}}(1^\lambda)\right] - 1 \in \mathrm{Negl} \ .$$

Bootstrapping



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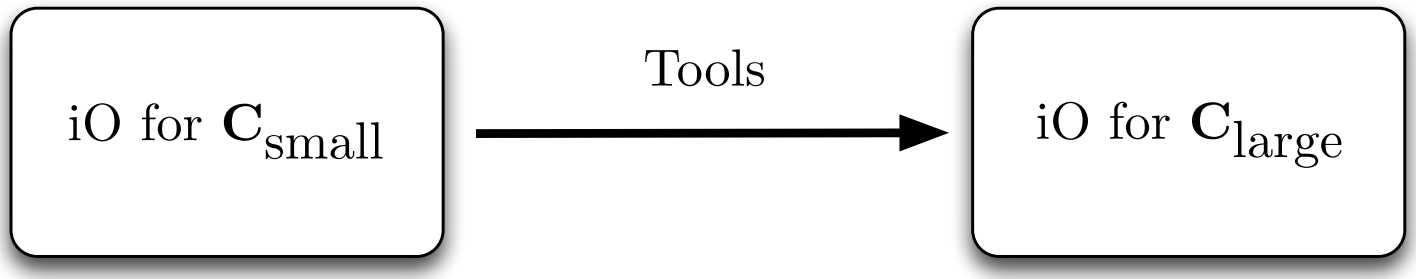

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$$\mathbf{C}_{\text{small}} = \mathbf{NC}^1 \quad \text{and} \quad \mathbf{C}_{\text{large}} = \mathbf{P/poly}$$

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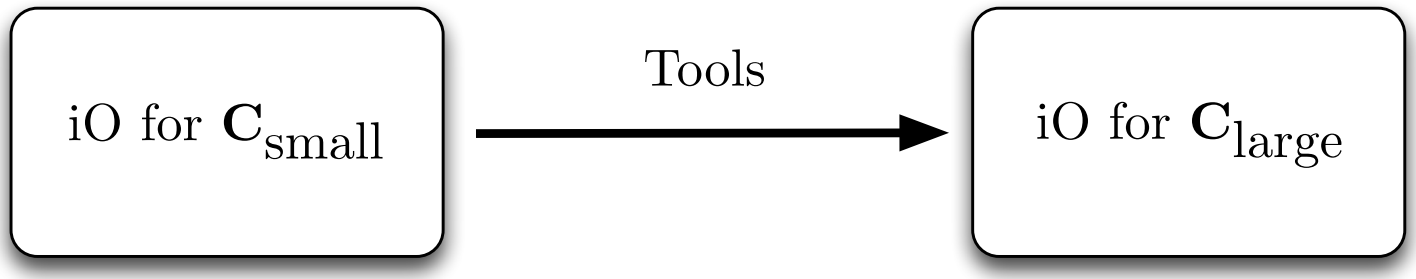

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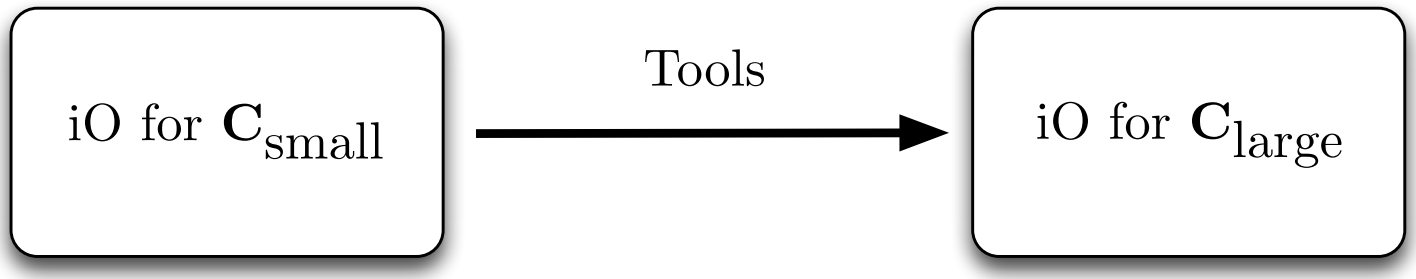

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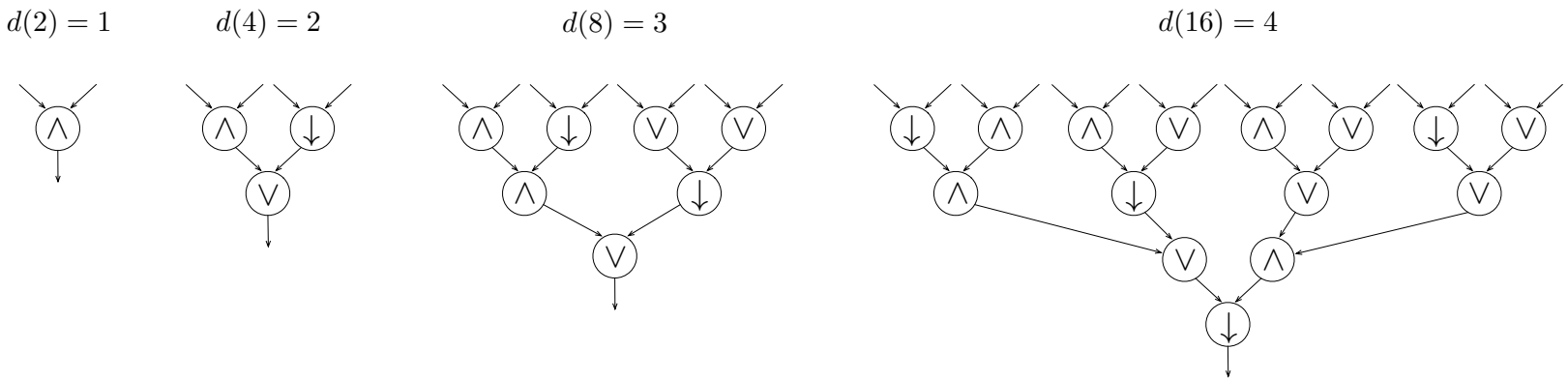
iO for $\mathbf{NC}^1$: See Mohammad's talk next!

Nick's Class

$\mathbf{NC}^1$ looks something like:



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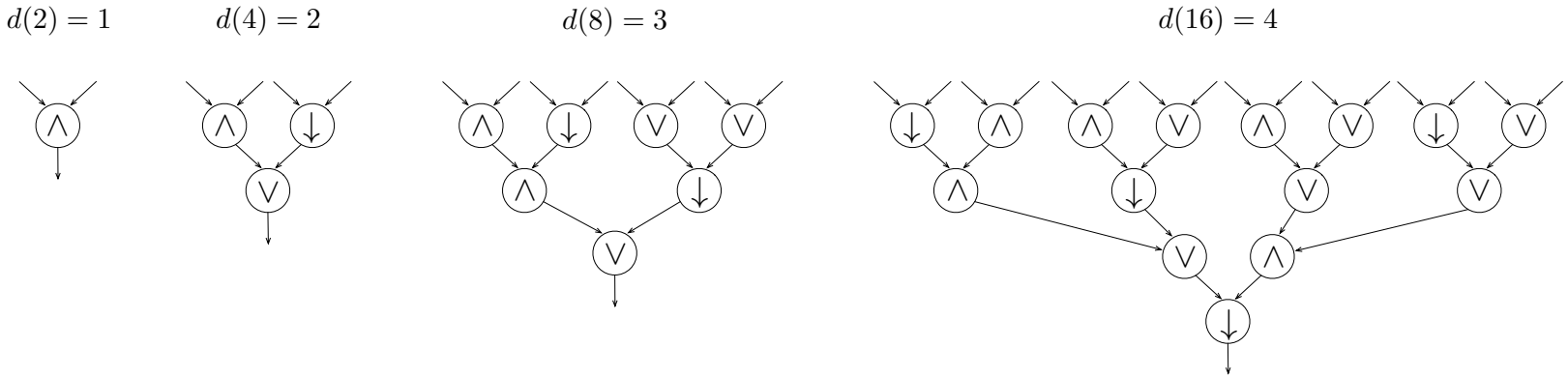

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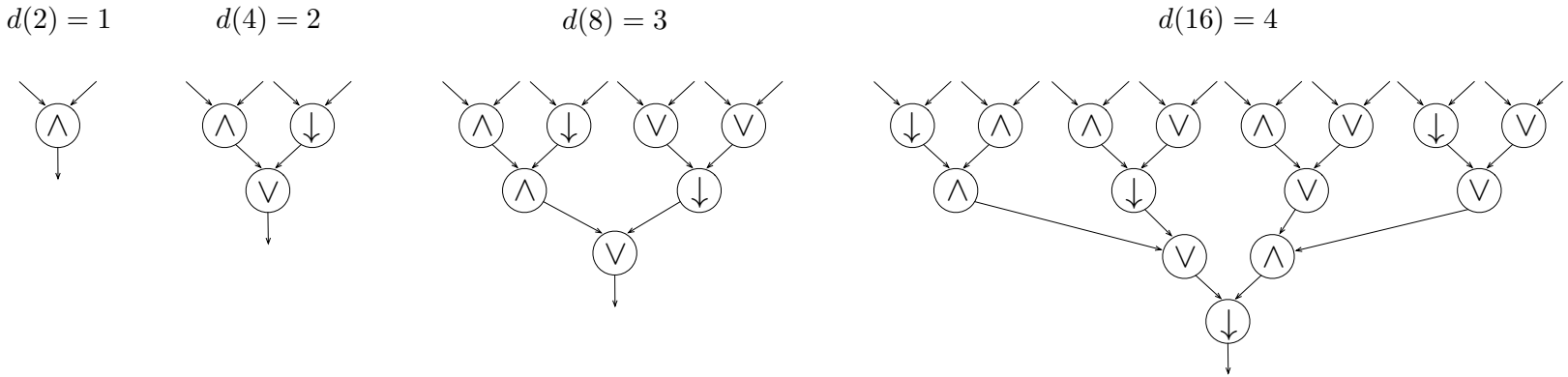
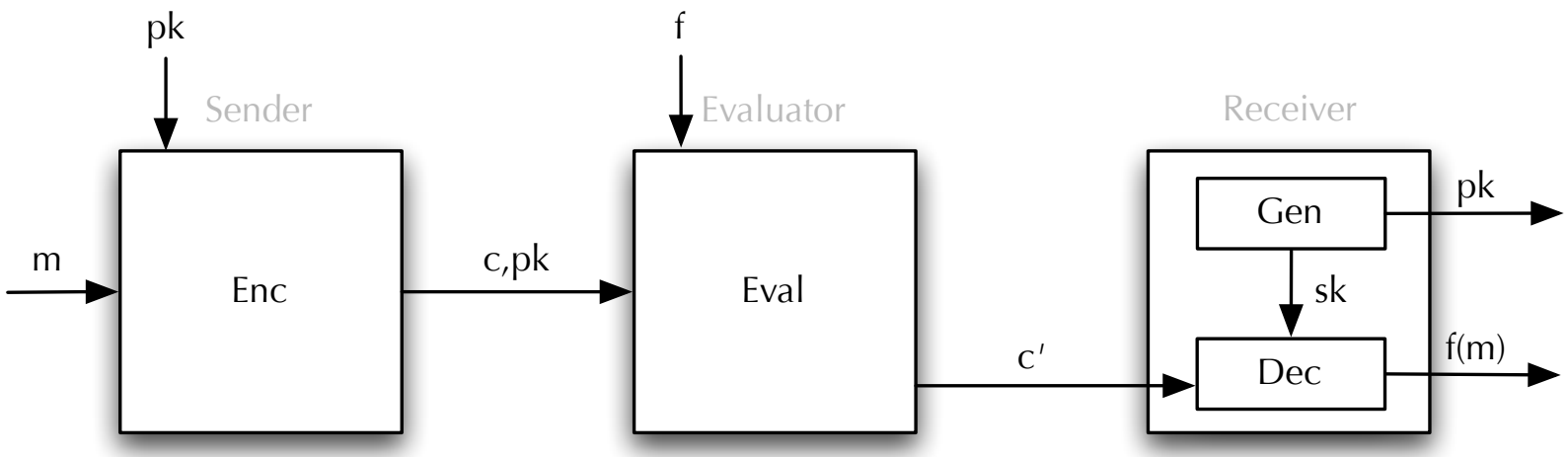


- $\mathbf{NC}^{\mathrm{i}} :=$ circuit families C_λ with constant fan-in/out gates whose depth grow like $\mathcal{O}(\log^i(\lambda))$ with input size λ.
- $\mathbf{P/poly} := \bigcup_{c \in \mathbb{N}} \{$ circuit families whose size grow like $\mathcal{O}(\lambda^c) \}$.

Fully Homomorphic Encryption (FHE)



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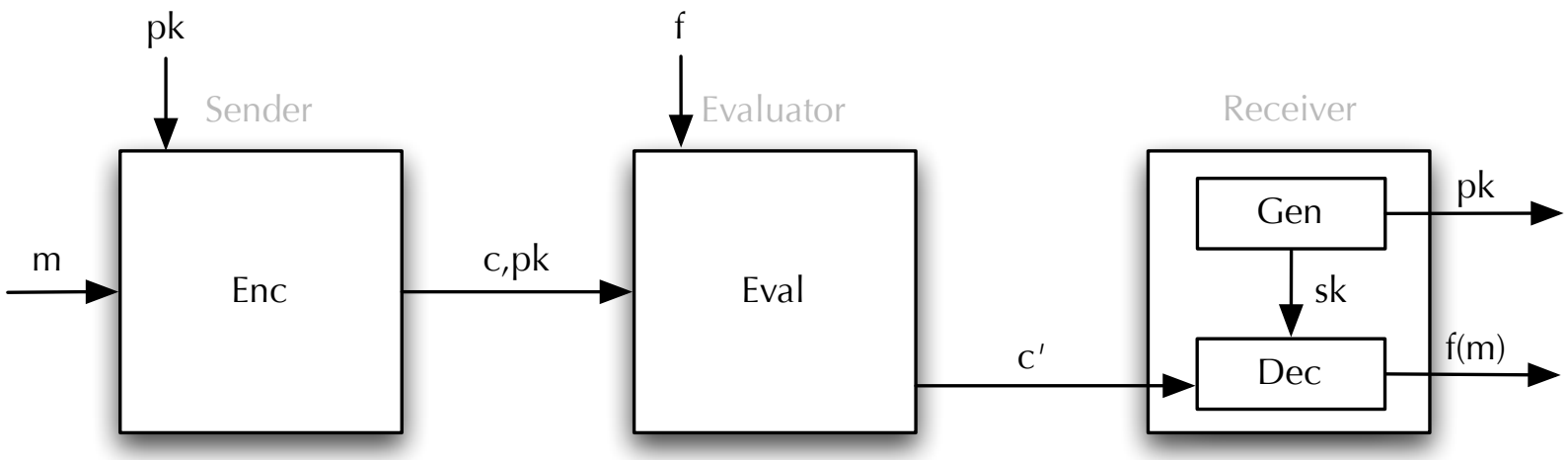

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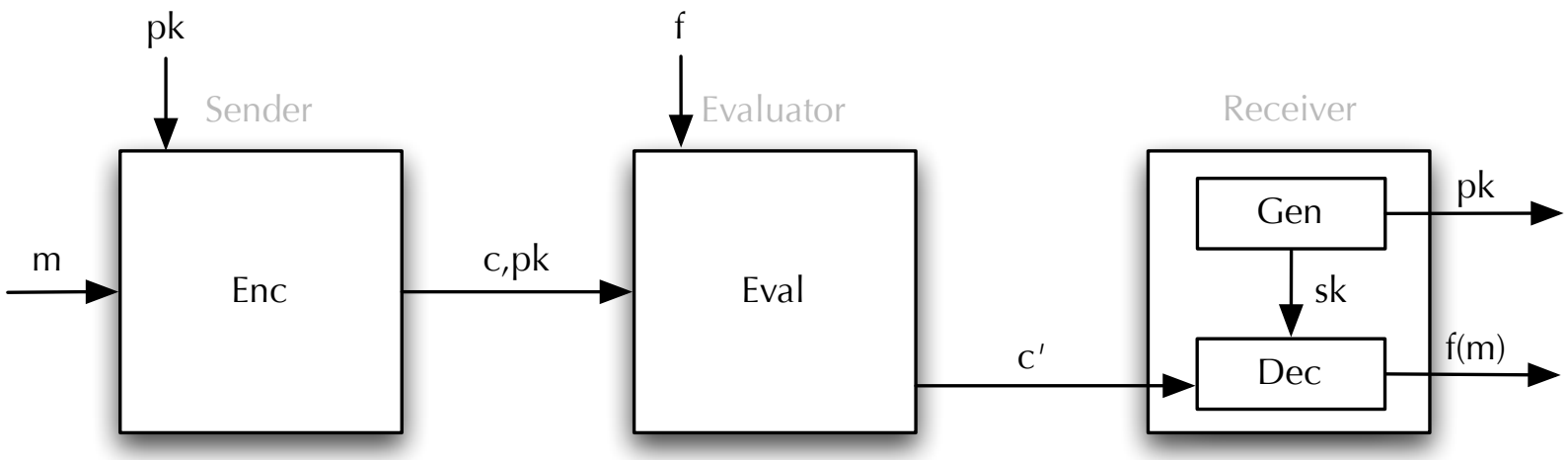
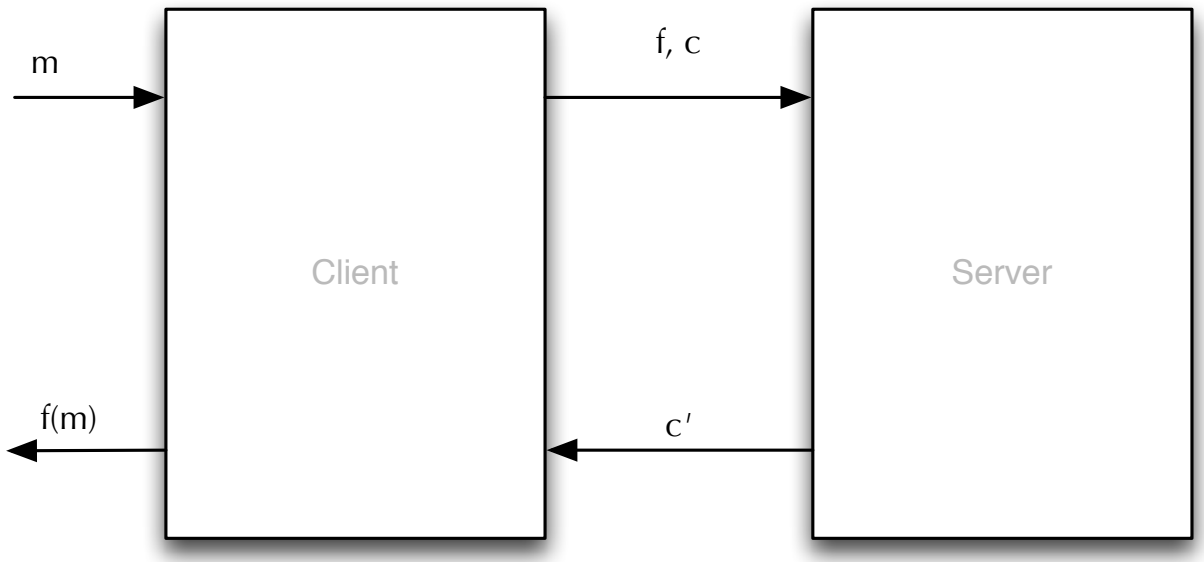


Scheme is:

Fully homomorphic if it supports all $f \in \mathbf{P/poly}$.
$L(\lambda)$-leveled homomorphic if it supports all $f \in \mathbf{NC}^{L(\cdot)}$.

Outsourcing of Computation



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This allows **switching** of keys in the proof.

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- Generate $\overline{P}$ as an $\underline{\mathbf{NC}^1 \text{ obfuscation}}$ of:

$$P[sk_1, G_1, G_2](m, E_1, E_2, \pi) := \text{if } \mathsf{V}((m, E_1, E_2), \pi) \text{ ret. } \mathsf{Dec}(sk_1, E_1)$$
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What's V? A low-depth proof of consistency that

$$\begin{aligned} \exists\,(G_1, G_2) \ : \quad & E_1 = \mathsf{Eval}(pk_1, \mathsf{UC}(\cdot, m), G_1) \\ \wedge \quad & E_2 = \mathsf{Eval}(pk_2, \mathsf{UC}(\cdot, m), G_2) \end{aligned}$$

Low-Depth Proof System (P, V) for Relation R

Perfect Completeness : For all (x, w) with $R(x, w) = 1$

$$\Pr[\pi \leftarrow_{\$} \mathsf{P}(x, w) : \mathsf{V}(x, \pi) = 1] = 1 \ .$$

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Construction:

$\mathsf{P}(x, w) :=$ List of all internal wire values of $R(x, w)$

$\mathsf{V}(x, \pi) :=$ Check if all wire values in π are consistent wrt. input x

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Details: Game $0 \longrightarrow$ Game 1

Given $(\mathcal{S}, \mathcal{A})$ that distinguishes Game 0 from Game 1, consider:

Algo. $\mathcal{B}^{\mathcal{S},\mathcal{A}}(pk_2)$:

$(C_0, C_1, st) \leftarrow\!\!\$\ \mathcal{S}(1^\lambda)$

Return (C_0, C_1) as messages.

Get ciphertext G_2 under pk_2.

$(sk_1, pk_1) \leftarrow\!\!\$\ \mathsf{Gen}(1^\lambda)$

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Return (P_0, P_1) (as two $\mathbf{NC}^1$ circuits).

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Algo. $\mathcal{B}^{\mathcal{A}}(\overline{P}, st)$:

$b' \leftarrow_{\$} \mathcal{A}(\overline{P}, st)$
return b'

Why is $\overline{\mathcal{S}}$ Eq-secure?

Need to show:

$$P[sk_1, G_1, G_2] \equiv P[sk_2, G_1, G_2]$$

Recall:

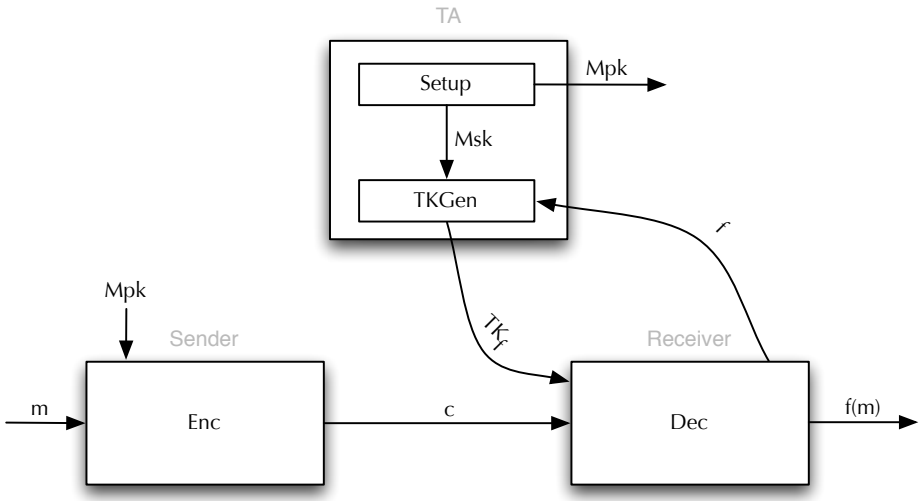
$$P[sk_1, G_1, G_2](m, E_1, E_2, \pi) := \text{if } \mathsf{V}((m, E_1, E_2), \pi) \text{ ret. } \mathsf{Dec}(sk_1, E_1)$$
$$\text{else return } \perp$$

This follows from:

- Proof π says E_1 and E_2 are correctly evaluated.
- The proof system is **perfectly** sound.
- FHE is **perfectly** correct.
- Circuits C_0 and C_1 are functionally equivalent.

Applications of iO

Functional Encryption (FE)



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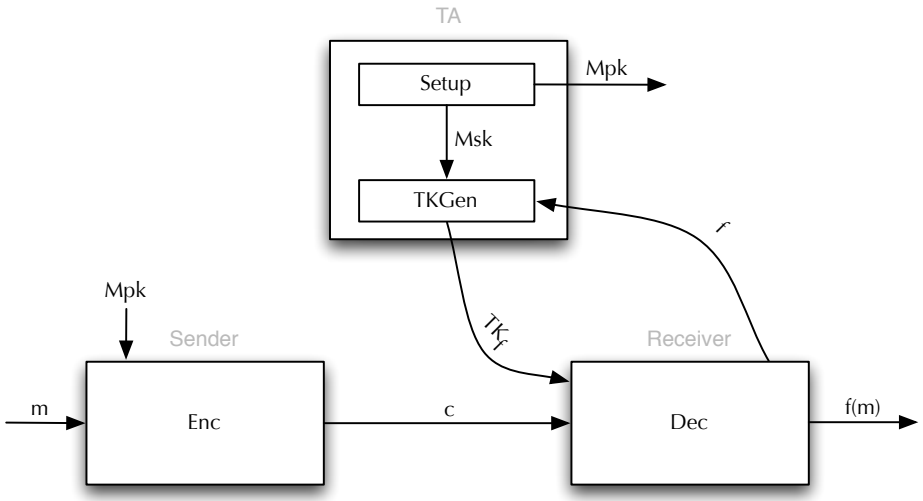

$$
\begin{aligned}
(Msk, Mpk) &\leftarrow\$ \quad \mathsf{Setup}(1^\lambda) \\
TK_f &\leftarrow\$ \quad \mathsf{TKGen}(f, Msk) \\
c &\leftarrow\$ \quad \mathsf{Enc}(\mathsf{m}, Mpk) \\
f(\mathsf{m}) / \perp &= \quad \mathsf{Dec}(c, TK_f)
\end{aligned}
$$

Attribute-Based Encryption

Fine-grained access to encrypted data.

$$f[\mathrm{P}](\mathrm{A}, m) := \begin{cases} m & \text{if } \mathrm{P}(\mathrm{A}) = 1 \ ; \\ \perp & \text{otherwise.} \end{cases}$$

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For instance:

- TA checks users' age $A \in [100]$ and issues secret key sk_A
- Policy can be set to

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For instance:

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- As a result, only adults can decrypt ciphertexts.

FE Security

Intuition: Requires that:

$$(\mathsf{Enc}(m_0), TK_{f_1}, \ldots, TK_{f_n}) \overset{c}{\approx} (\mathsf{Enc}(m_1), TK_{f_1}, \ldots, TK_{f_n})$$

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$b \leftarrow\$\ \{0,1\}$
$c \leftarrow\$\ \mathsf{Enc}(mpk, m_b)$
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Advantage:

$$\mathbf{Adv}^{\text{ind-cpa}}_{\mathsf{FE},\mathcal{A}}(\lambda) := 2 \cdot \Pr\left[\text{IND-CPA}^{\mathcal{A}}_{\mathsf{FE}}(1^\lambda)\right] - 1$$

The iO-to-FE Transform

Intuition:

Setup(1^λ):	$msk = sk$ and $mpk = pk$.
Enc(mpk, m):	Encrypt using PKEnc(pk, m).
TKGen(msk, f):	$TK_f =$ Obfuscation of $f($PKDec$(sk, \cdot))$.
Dec(TK_f, c):	Run TK_f on c.

To "switch" keys/messages in the reduction, attach proofs of consistency.

FE Construction – Details

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And a **NIZK proof** π of well-formedness:

$$\exists\,(m, r_1, r_2) \;: E_1 = \mathsf{PKEnc}(pk_1, m; r_1)$$
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$\mathsf{Dec}(c, TK_f)$: Run TK_f on $c = (E_1, E_2, \pi)$.

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Computational ZK : There is a $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$ st. for all ppt $\mathcal{A}$ and all $x \in R$

$$\Pr[\sigma \leftarrow_{\$} \mathsf{K}(1^{\lambda}); \pi \leftarrow_{\$} \mathsf{P}(\sigma, x, w) : \mathcal{A}(\sigma, x, \pi) = 1]$$
$$\overset{c}{\approx} \Pr[(\sigma, \tau) \leftarrow_{\$} \mathcal{S}_1(1^{\lambda}, x); \pi \leftarrow_{\$} \mathcal{S}_2(\sigma, x, \tau) : \mathcal{A}(\sigma, x, \pi) = 1]$$

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Proof of Security

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    By the rules of FE game $f_{i+1}(m_0) = f_{i+1}(m_1)$.

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