The Provable Security Methodology

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Mathematical Aspects of Computer Science – Foundations of Cryptography

متدولوژیِ امنیت قابل اثبات

پویا فرشیم

دانشگاه کوئینز بلفاست – اکول نرمال سوپریور

جنبه‌های ریاضی علوم کامپیوتری – مبانی رمز

Unreasonable Effectiveness of Mathematics

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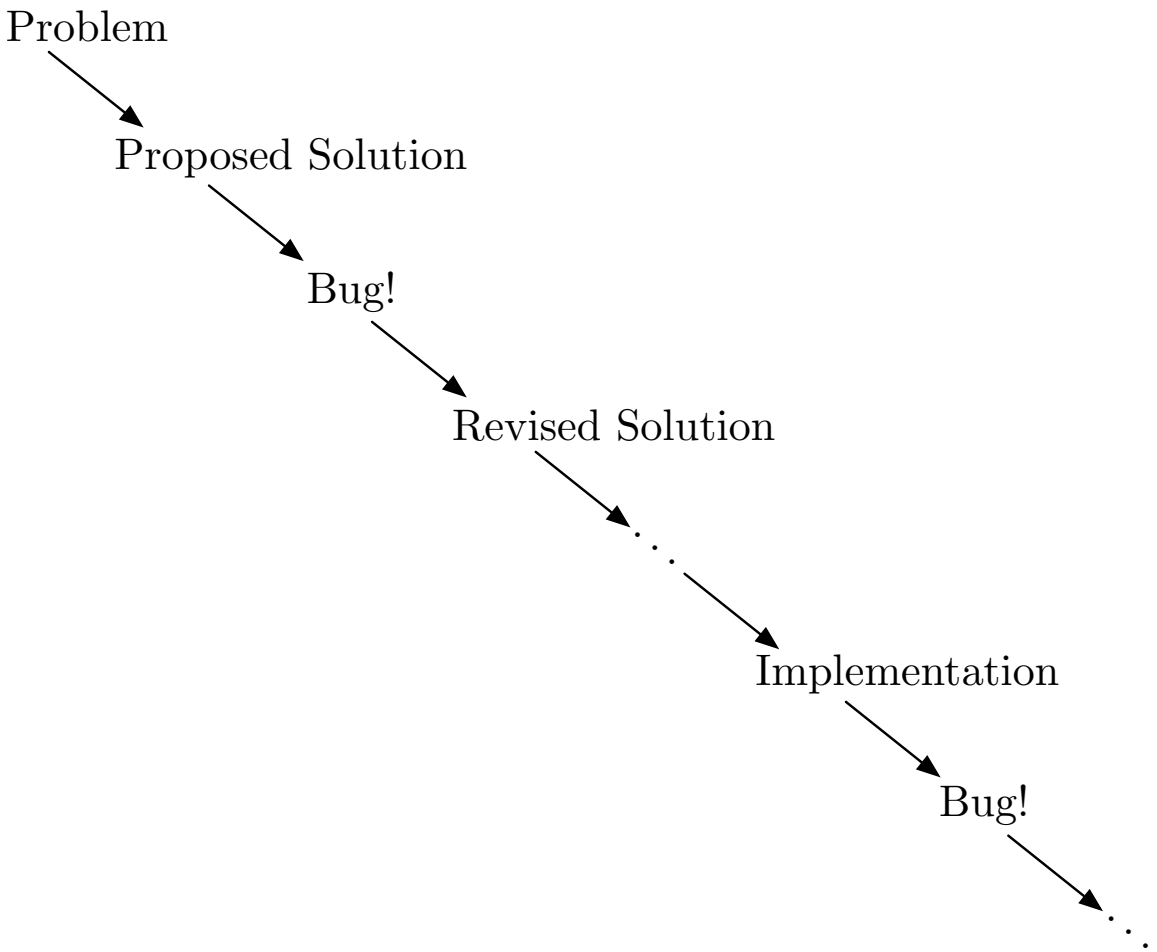
1. E.P. Wigner. The unreasonable effectiveness of mathematics in the natural sciences. *Com. in Pure and Applied Mathematics*, 1960.
2. R.W. Hamming. The unreasonable effectiveness of mathematics. *The American Mathematical Monthly*, 1980.

Cryptography, *n*

The Oxford English Dictionary defines:

> **cryptography**, *n.* **1.** The art or practice of writing in code or cipher;

Ad Hoc Protocol Deign



Source: Bellare & Rogaway. Introduction to Modern Cryptography. 2005.

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> the science of encryption; the branch of cryptology concerned with this (cf. cryptanalysis *n.*). More generally: the study of codes and ciphers; cryptology.

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A broader definition that's perhaps better suited for us:

> **cryptology**: the scientific/mathematical study of systems in the presence of adversarial behavior.

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(4) Proof that an instance of (1) is secure wrt. definition (2) relative to assumptions (3).

See "Post-Modern Cryptography" [Gol06] for a (vehement!) defense.

The Role of Definitions

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2. $\underline{\mathsf{F}(fk, x)}$: is deterministic; given fk and $x \in \mathsf{D}_{fk}$ outputs $y \in \mathsf{R}_{fk}$.
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- Permutations, injective functions, compressing functions, etc.

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A cryptosystem is called secure if

Any adversary of specified resources has small advantage (according to the metric) in achieving the specified break.

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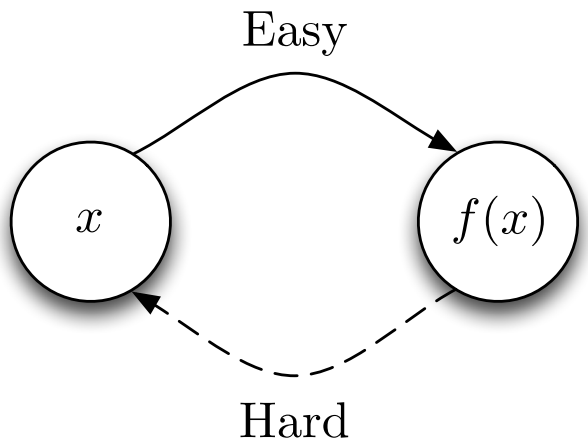

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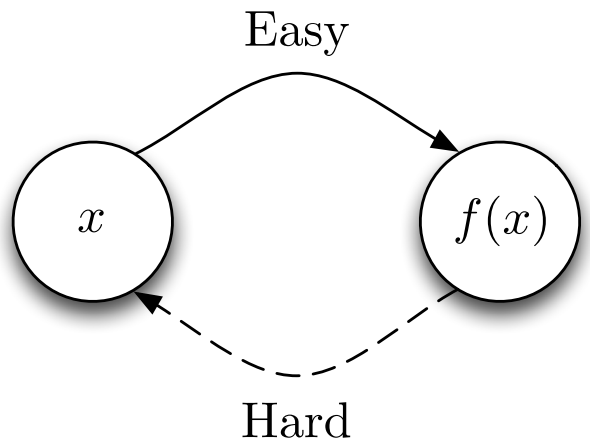

Formalization involves multiple choices:

- Computing **a** pre-image vs. **the** pre-image: Is $x \mapsto 0$ useful?
- How is $f(x)$ sampled? Is it sampled at all?
- Some pre-image can always be found in exponential time.
- And can always guess x with nonzero probability.
- How "small" a probability do we want?

Negligible Functions

A negligible function $\mu : \mathbb{N} \longrightarrow \mathbb{R}$ approaches zero faster that $1/\mathrm{poly}(\lambda)$ for any $\mathrm{poly}(\lambda)$.[1] In other terms

$$\mu(\lambda) \in \lambda^{-\omega(1)} \ .$$

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Define

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Require:

$$\forall \text{ ppt TMs } A \ : \ \mathbf{Adv}^{\mathrm{ow}}_{\mathsf{FF},A}(\lambda) \in \mathrm{Negl} \ .$$

The “Right” Notion?

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Can we relate these notions?

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- Ultimately: How accepting the community is, and how fruitful a definition turns out to be.

(3) Assumptions [GK15]

Definition (Crypto Assumption)

A (search) cryptographic assumption is an interactive protocol

$$\langle \mathsf{C}(1^\lambda), A(1^\lambda) \rangle$$

between a challenger C and an adversary A that terminates with C outputting T/F. The assumption holds in the average case if for all ppt A

$$\mathbf{Adv}_{\mathsf{C},A}(\lambda) := \Pr[\langle \mathsf{C}(1^\lambda), A(1^\lambda) \rangle] \in \mathrm{Negl} \ .$$

It holds in the worst case if for all non-uniform pt A

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When C is ppt, we call the assumption **falsifiable** [Nao03].

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To what extent are these existentially equivalent?

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What about other primitives?

Can we base average-case hash functions on worst-case hash functions?

Universality of OWF

Theorem (Universal OWF [Gold01])

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Formalize universality. Which primitives are universal?

(4) Security Results

Results in provable security often look like this:

Theorem

If scheme D is secure wrt. its definition of security, then construction C is secure wrt. its definition of security. More precisely, for any $t(\lambda)$-time A, there is a $t'(\lambda)$-time adversary B such that:

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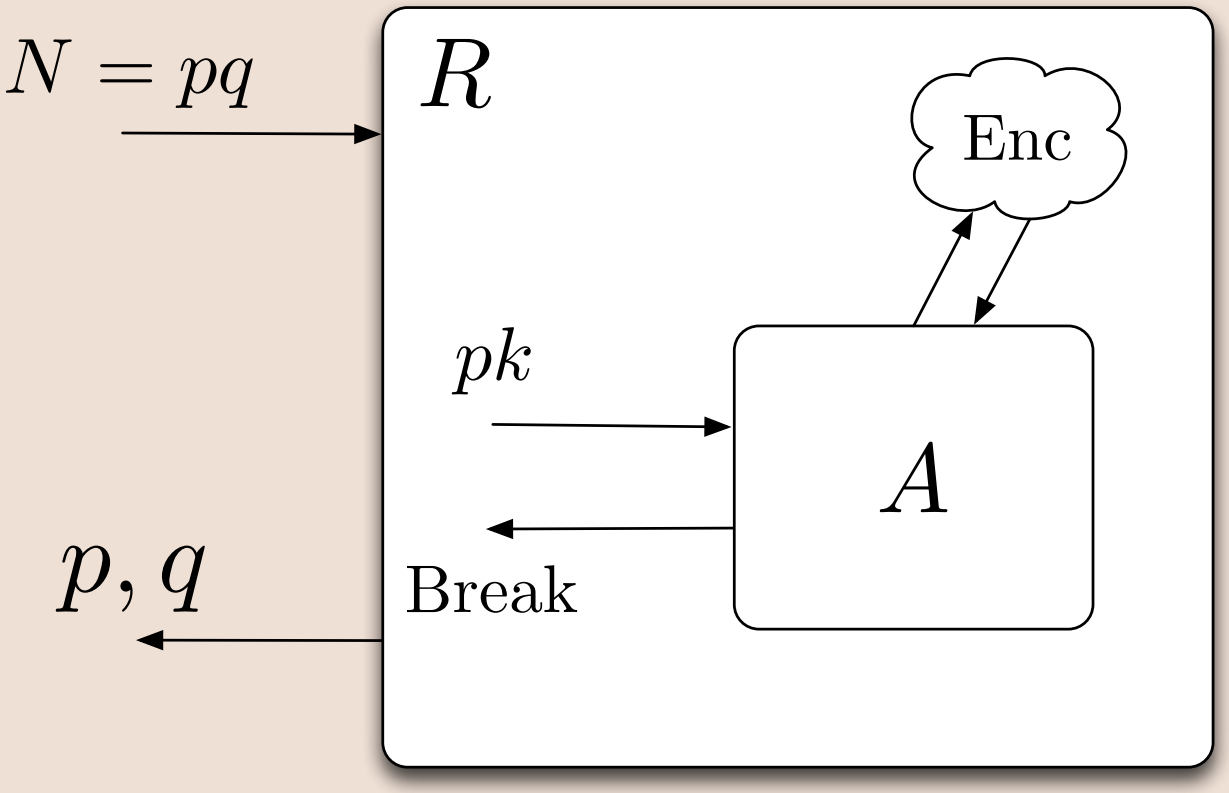
Guides us how to set key sizes.

Proofs via Reductions

Proof.



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Caveats:

- Provable security never proves security in an absolute sense.
- Security is always relative to the assumptions and the model.
- Definitions, assumptions, protocols and proofs change!

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Require:

$$\forall \text{ ppt } A \ : \ \mathbf{Adv}^{\mathrm{cr}}_{\mathsf{FF},A}(\lambda) := \mathsf{Pr}[\mathrm{CR}^{A}_{\mathsf{FF}}(1^\lambda)] \in \mathrm{Negl} \ .$$

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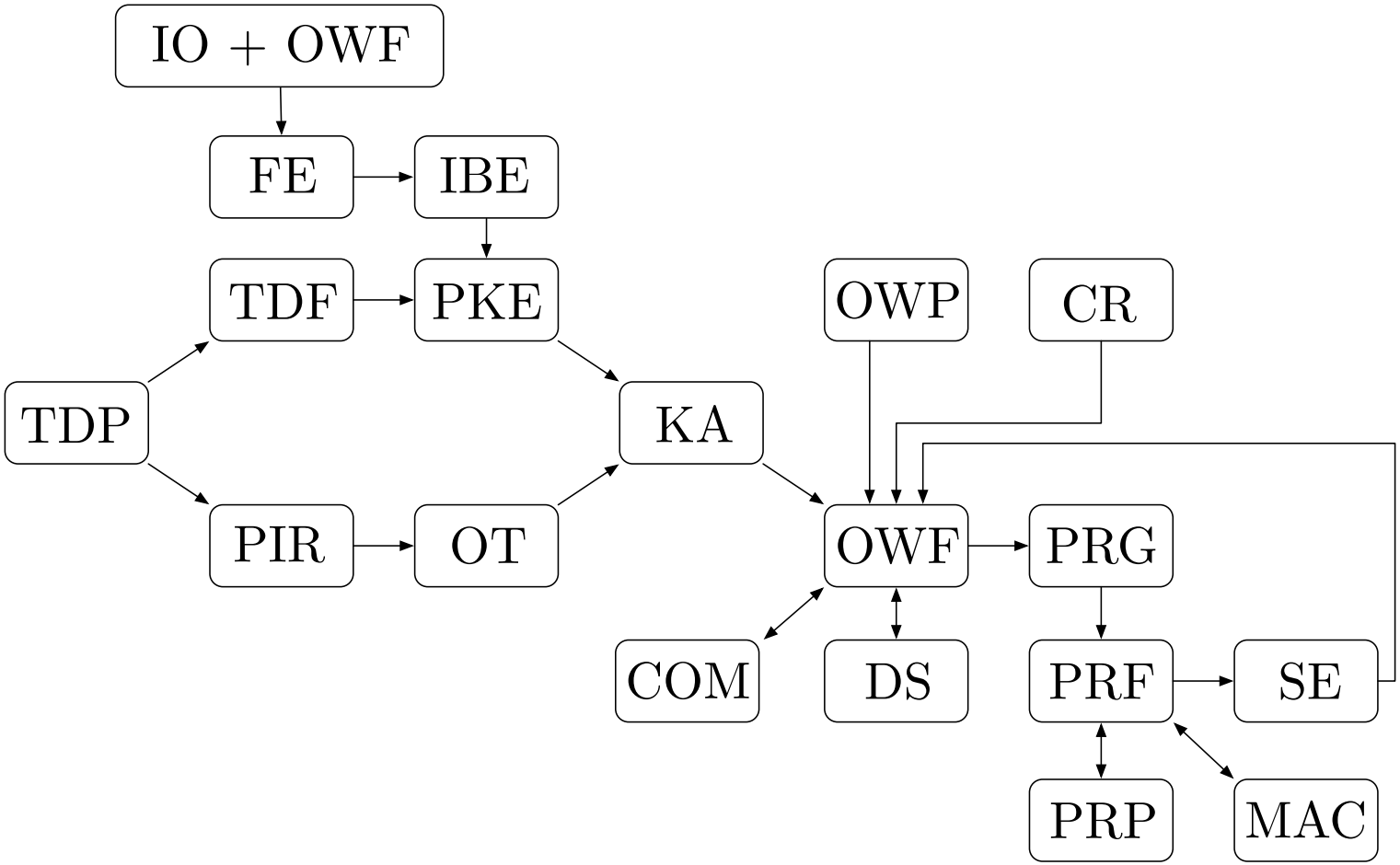
OWF is (strictly) weaker than CRF as an assumption.

Landscape of Primitives



Thanks!