Domain Theory and Differential Calculus

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- Generalised Cauchy Riemann equations for Lipschitz maps

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- Continuous Scott domains are characterised by the above property.

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- Suppose $f : [0, 1] \to \mathbb{R} \subseteq \mathbf{I}\mathbb{R}$ is continuous.
- $[0, 1] \subseteq \mathbf{I}[0, 1]$ is a dense subset.
- The continuous maximal extension of f is

$$f^* : \mathbf{I}[0, 1] \to \mathbf{I}\mathbb{R}$$

$$f^*(a) = \{f(x) : x \in a\}$$

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Partial Extensions: Example

- Consider $S : \mathbb{R} \to \mathbb{R} \subseteq \mathbf{I}\mathbb{R}$ with

$$S(x) = x - \lfloor x \rfloor$$

- S has a discontinuity at each integer point.
- $S^* : \mathbb{R} \to \mathbf{I}[0, 1]$ is given by

$$S^*(x) = x - \lfloor x \rfloor \text{ if } x \notin \mathbb{Z}$$

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- **Example:** $f : \mathbb{R} \to \mathbb{R} : x \mapsto |x|$ has $\partial f(0) = [-1, 1]$.

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$$\partial f(x) = \text{Conv}\{\lim f'(x_i) : x_i \to x,\ x_i \notin \Omega_f\} \in \mathbf{C}(\mathbb{R}^{m\times n}),$$

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- It was later proved $\partial f(x)$ has an intrinsic value independent of any null set used in its definition.

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- Basic Scott (or upper)open set on $\mathbf{C}(\mathbb{R}^{m\times n})$:

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- **Theorem**
$$(f')^* = \partial f$$

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- The **L-derivative** of f at $x \in U$ is the Scott continuous map:

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- Define $f \in \delta(a, b)$ for $a \subseteq U$ open and $b \in \mathbf{C}((\mathcal{H}^*)^m)$ if

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- For $\mathbb{K} = \mathbb{R}$ this works for any Banach space. For $\mathbb{K} = \mathbf{C}$?

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- A continuous Scott domain V_{mn} for Lipschitz maps of type $U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is constructed as a subset of $(U \to \mathbf{I}\mathbb{R}^m) \times (U \to \mathbf{C}(\mathbb{R}^{m\times n}))$ with

$$(f, g) \in V_{mn} \text{ if } \exists h : \mathbb{R}^n \to \mathbb{R}^m .\, f \sqsubseteq h \,\&\, g \sqsubseteq \mathcal{L}h$$

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- The domain V_{11} used to develop a **programming language** equipped with a real number data type in which computable Lipschitz maps and their derivatives can be numerically evaluation at any computable number. (De Gianantonio and AE 13).

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- Let the **L-topology** be the refinement of the D-topology and the sup norm topogy on $\text{Lip}(U)$.

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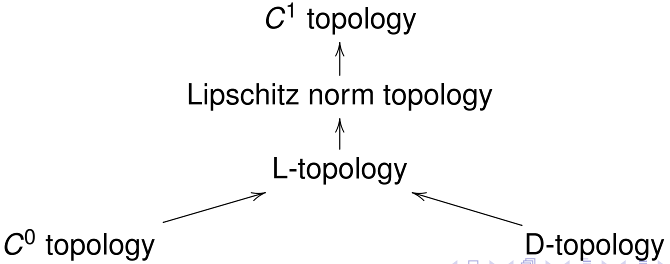
- Let the **L-topology** be the refinement of the D-topology and the sup norm topogy on $\mathrm{Lip}(U)$.

- L-topology is weaker than the Lipschitz norm topology:



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- **Corollary.** The theorem characterises the L-topology.

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- Then, $\lim_{k\to\infty} f_k = f$ in the L-topology. Implications?

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- **Theorem** (AE, LICS 15) Clarke's gradient operator is the extension of the derivative operator wrt the densely injective property:

$$D^* = \partial$$

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- This result, in some other notation, is in Riemann's thesis.

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- **Theorem.** (Cauchy Riemann equations for Lipschitz maps)

$$\mathcal{L}f(z) = \mathrm{conv} \bigcup \left\{ \mathrm{D}(\overline{v}, i\,\overline{w}) : \begin{bmatrix} v \\ w \end{bmatrix} \in \mathcal{L}\hat{f}(x, y) \right\}$$

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$$\mathcal{L}\hat{f}(x,y)\subseteq(\mathcal{L}V(x,y)\times\mathcal{L}W(x,y))\in\mathbb{C}(\mathbb{R}^{2\times 2})$$

- $\mathrm{D}(a,b)\subseteq\mathbb{C}$: disk with a diameter with ends at $a,b\in\mathbb{C}$.
- **Theorem.** (Cauchy Riemann equations for Lipschitz maps)

$$\mathcal{L}f(z)=\operatorname{conv}\bigcup\left\{\mathrm{D}(\overline{v},i\,\overline{w}):\begin{bmatrix}v\\w\end{bmatrix}\in\mathcal{L}\hat{f}(x,y)\right\}$$

- If f is analytic, $\mathcal{L}V=V'$ and $\mathcal{L}W=W'$ are single points:

$$\mathcal{L}f(z)=f'(z)=\overline{V'}=\frac{\partial V}{\partial x}-i\frac{\partial V}{\partial y}=i\,\overline{W}=\frac{\partial W}{\partial y}+i\frac{\partial W}{\partial x}.$$

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- i.e., a single disk $\mathrm{D}(\overline{V'},i\,\overline{W'})$ with a single point $\overline{V'}=i\,\overline{W'}$.

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- If $f(z) = |z|$, then $\mathcal{L}V(0,0)$ is the unit disk at 0.

Thus: $$\mathcal{L}f(z) = \begin{cases} \mathrm{D}(e^{-i\theta}, 0) & \text{for} \quad z \neq 0 \\ \mathrm{D}(-1, 1) & \text{for} \quad z = 0 \end{cases}$$

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\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} & x \leq 0 \;\&\; y \geq |x| \\
\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} & x \leq 0 \;\&\; 0 \leq y \leq |x| \\
\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} & x \leq 0 \;\&\; 0 \geq y \geq x \\
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- $$\text{Left set} = \mathcal{L}f(0) = \text{conv}\bigcup\left\{\mathrm{D}(\overline{v}, i\,\overline{w}) : \begin{bmatrix} v \\ w \end{bmatrix} \in \partial\hat{f}(0,0)\right\} \subsetneq$$

$$\text{Right set} = \text{conv}\bigcup\{\mathrm{D}(\overline{v}, i\,\overline{w}) : v \in \mathcal{L}V(0,0), w \in \mathcal{L}W(0,0)\}$$

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- Thus there exists $C \in \mathbf{C}(\mathbb{C})$ such that $S(v) = \sup(C \cdot v)$.
- We define $\int_p g := C$.

THANK YOU