Domain Theory and Integral Calculus

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  - $\nu(\emptyset) = 0$
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  - $\nu(O_1) + \nu(O_2) = \nu(O_1 \cup O_2) + \nu(O_1 \cap O_2)$
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- A **simple valuation** is of the form $\nu = \sum_{i=1}^{n} r_i \delta_{y_i}$ where $y_i \in Y$ for $1 \leq i \leq n$ and $r_i > 0$ with

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- For two normalised simple valuations

$$\mu_1 = \sum_{b \in B} r_b \delta_b, \qquad \mu_2 = \sum_{c \in C} s_c \delta_c$$

$\mu_1 \sqsubseteq \mu_2$ iff,

$$\forall b \in B, \forall c \in C, \exists t_{b,c} \geq 0.$$

$$\sum_{c \in C} t_{b,c} = r_b \qquad \sum_{b \in B} t_{b,c} = s_c$$

and $t_{b,c} \neq 0$ implies $b \sqsubseteq c$.

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- The closures of the elements in B give a countable basis for $\mathbf{U}X$.
- Thus, $\mathbf{P}^1(\mathbf{U}X)$ is an ω-continuous domain with a basis of simple valuations of the form

$$\sum_{i=1}^{n} r_{\overline{b_i}} \delta_{\overline{b_i}}, \text{ with } b_i \in B \text{ for } 1 \leq i \leq n$$

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- $e : \mathbf{M}^1 X \to \mathbf{P}^1(\mathbf{U}X)$ with $e(\mu) = \mu \circ s^{-1}$ is an embedding into the set of maximal elements of $\mathbf{P}^1(\mathbf{U}X)$.

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- Thus μ, identified with $e(\mu)$, can be approximated by the sup of an increasing sequence of simple valuations:

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- This provides the basic idea for the R-integral.
- The map e is onto the set of maximal elements of $\mathbf{P}^1(\mathbf{U}X)$ (Lawson 95).

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- Then,

$$e(\lambda) = \sup_{n \geq 0} \nu_{P_n}$$

R-integration on a compact metric space X

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- For each simple valuation $\nu = \sum_{c \in C} r_c \delta_c \in \mathbf{P}^1(\mathbf{U}X)$, define the lower and upper sums,

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- $S^\ell(f,\nu_i) \leq S^\ell(f,\nu_{i+1}) \leq S^u(f,\nu_{i+1}) \leq S^u(f,\nu_i)$ for all $i \geq 0$
- If the set of discontinuities of f has μ measure zero then

$$\sup_{i \geq 0} S^\ell(f,\nu_i) = \int f \, d\mu = \inf_{i \geq 0} S^u(f,\nu_i)$$

where $\int f \, d\mu$ is the Lebesgue integral of f wrt μ.

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$$L_{f,P} = \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{x \in [x_{i-1}, x_i]} f(x)$$

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- Thus,

$$S^{\ell}(f, \mu_P) = L_{f,P}, \qquad S^{u}(f, \mu_P) = U_{f,P}$$

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- Continuous Scott domains are characterised by the above property.

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- Since $\mathbb{R} \subset \mathbf{I}\mathbb{R}$, we have a continuous map

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- **Question:** Find an explicit expression for

$$\int_R^* f \, d\nu := \int_R^* (f, \nu)$$

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- Define $\int(f, \nu) = [S^\ell(f^-, \nu), S^u(f^+, \nu)]$.
- **Proposition.** If $f = \sup_{i\geq 0} f_i$ and $\nu = \sup_{i\geq 0} \nu_i$ are sups of step functions f_i and simple valuations ν_i for $i \geq 0$, then
$$S^\ell(f^-, \nu) = \sup_i S^\ell(f_i^-, \nu_i), \quad S^u(f^+, \nu) = \inf_i S^u(f_i^+, \nu_i)$$

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- **Proposition.** If $f = \sup_{i\geq 0} f_i$ and $\nu = \sup_{i\geq 0} \nu_i$ are sups of step functions f_i and simple valuations ν_i for $i \geq 0$, then

$$S^\ell(f^-, \nu) = \sup_i S^\ell(f_i^-, \nu_i), \quad S^u(f^+, \nu) = \inf_i S^u(f_i^+, \nu_i)$$

- Get a continuous extension $\int : (X \to \mathbf{I}\mathbb{R}) \times \mathbf{P}^1(\mathbf{U}(X)) \to \mathbf{I}\mathbb{R}$.

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- **Theorem**

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- The map $x \mapsto \{x\} : X \to \mathbf{B}(X)$ embedds X onto the set of maximal elements of $\mathbf{B}(X)$.

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- Thus, $C_c^0(\mathbb{R}^n)$ is dense in $\mathbf{B}(L^1(\mathbb{R}^n, \mu))$.
- The map

$$\int_R (-)\, d\mu : C_c^0(\mathbb{R}^n) \subseteq \mathbf{B}(L^1(\mathbb{R}^n, \mu)) \to \mathbb{R} \subseteq \mathbf{I}\mathbb{R}$$

  is continuous.
- Thus, we can apply the extension theorem.

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- By the extension theorem, there is a maximal continuous extension of f:

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- More generally if $B \in \mathbf{B}(L^1(\mathbb{R}^n, \mu))$, then

$$\int_R^* B\, d\mu = \left\{ \int f\, d\mu : f \in B \right\}$$

A new way of obtaining the Lebesgue integral

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- Thus, using the R-integral we can obtain the Lebesgue integral in a new way using the generalised Riemann integral of continuous functions that approximate f.

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- In particular if $a = b$, then the path integral is always zero independent of the closed path p.

Weak topology for Lipschitz maps (AE 07)

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- The **D-topology** on the space Lip(U) of Lipschitz maps $U \subseteq \mathbb{K}^n \to \mathbb{K}$ is the weakest topology such that

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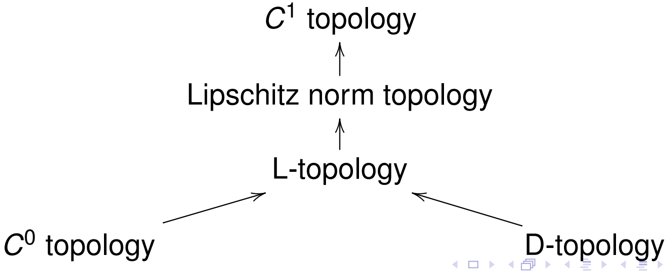
- L-topology is weaker than the Lipschitz norm topology:



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- Thus: $f_k(x) = \int_U f(x) T_k(y - x)\, dx$ and $f_k \in C^\infty(U)$.
- Then, $\lim_{k\to\infty} f_k = f$ in the L-topology. Implications?

Extension of Green's Theorem (I)

- Fix a piecewise C^1 map $p : [0,1] \to U$ and define

$$D_p : C^1(U) \cap \mathrm{Lip}(U) \to ([0,1] \to \mathbf{I}\mathbb{R})$$

$$D_p(g) = \lambda t.\, g'(p(t)) \cdot p'(t),$$

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- we have $\int_{[0,1]} f_n'(p(t) \cdot p'(t)\, dt = f_n(p(1)) - f_n(p(0))$ and $\lim_{n\to\infty} f_n(p(1)) - f_n(p(0)) = f(p(1) - f(p(0))$.

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- Extension theorem: Get interval version Green's theorem:

$$f(p(1)) - f(p(0)) \in \left(\int_{[0,1]}\right)^* \circ (D_p)^*(f)$$

THANK YOU